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Modeling of Stability of Electrostatic and Magnetostatic Systems

by Michael Grinfeld and Pavel Grinfeld

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14. ABSTRACT
Electromagnetic systems undergo a variety of different instabilities. A broad class of those instabilities shows up in the static or quasi-static regime. For special, highly symmetric shapes (spheres or flat layers), the parameters, corresponding to the occurrence of those instabilities, can be investigated rather easily using analytical methods. Slightly postcritical regimes require much more sophisticated and cumbersome—but still analytical—calculations. Strongly postcritical regimes always require very heavy computer work and rather sophisticated methodologies. In this report we present a computational algorithm allowing one to address some strongly nonlinear regimes. The algorithm is based on the variational formulation of the problems under study and presented on the Rayleigh instability of a charged liquid conductor phenomenon.

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1. Introduction

The systems carrying electric charges are widespread in nature and applications.

Variational principles and methods play a key role in modeling these systems, and they have been developed for a couple of centuries.[1–5] The issues of stability and instability of electrostatic, magnetostatic, and quasi-static systems are extremely important in multiple electrostatic systems (e.g., Earnshaw theorem or the Rayleigh surface instability) and in multiple magnetostatic systems (e.g., the well-known surfacial patterning in ferrofluids demonstrated in Fig. 1).

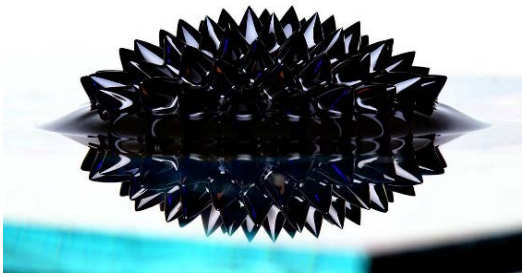

Fig. 1 Ferrofluid on a reflective glass plate under the influence of a strong/steep magnetic field. (Reprinted with permission of Gregory Maxwell, Wikipedia: https://en.wikipedia.org/_wiki/ File: Ferrofluid_Magnet_under_glass.jpg#/.)

Therefore, it is quite surprising that the key instruments in the analysis of stability (e.g., second-energy variation) are essentially underdeveloped. Also, various variational approaches, closely associated with the first- and second-energy variation, require quite sophisticated mathematical techniques to handle them.

Variational approaches are of key importance in computational implementations of the models. It happens because of the a priori guaranteed mathematical self-consistency of these methods and open possibilities of using powerful numerical schemes based on minimization and optimization. Nonetheless, even a quick glance at the literature shows that the variational approach is definitely underdeveloped. In particular, in terms of calculus of variations, all of the studies are based on the analysis of the first-energy variations (i.e., analysis of ponderomotive forces and conditions of equilibriums.) The analysis of the second variation, the cornerstone of the stability investigation, is not even touched in those famous publications. Moreover, the very expressions of the second-energy variations still remain unknown (Grinfeld[5–7] and references therein are very rare exceptions to this rule).

All of the discussed issues lead to the conclusion that numerical modeling remains the only realistic chance of exploring various morphological patterns appearing at the late stage of destabilization.

In the following, we present an iteration scheme of possible numerical analysis of the poststability patterns. The approach is based on the variational formulation of the model system under study. The presentation is limited to discussion of the classical Rayleigh instability of a charged conducting liquid drop. The approach permits extension to several other systems, including those accumulating and carrying free electric charges not only on the surface, but also inside the bulk, as well as the pyroelectric crystals growing on rigid substrates, patterns of ferromagnetic liquids, and the like. To make the presentation as simple as possible, we limit ourselves to incompressible substances and ignore polarization effects.

2. Electrocapillary Instabilities

Rayleigh first predicted[8] that an isolated charged conducting drop is morphologically unstable if the electric charge is sufficiently great.

The intuitive explanation (not the professional proof) for the Rayleigh instability is rather straightforward. The electrons repel each other and try to stay as far as possible from each other. Therefore, they create the forces deforming the drop. The assumption of the liquid incompressibility does not allow the spherical drop to change its radius. Therefore, the available shape modifications are not radial. One of the simplest nonradial modifications is the spheroidal perturbation, as shown in Fig. 2.

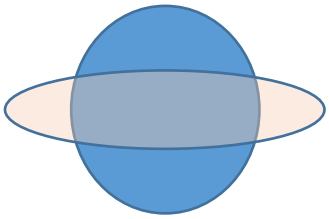

Fig. 2 Rayleigh instability of a charged conducting incompressible liquid. For a sufficiently high charge or small surface energy, the spherical drop becomes unstable with respect to spheroidal perturbations.

From the energetic point of view, we would say that electrons try to diminish the accumulated electrostatic energy. To put it another way, the electrostatic energy of the system can be lowered by letting the drop deviate from spherical shape. Such deformation, however, is opposed by surface tension since the minimum surface energy is attained when the shape is spherical.

The overall instability criterion, then, reflects the balance between these forces. If R is the radius of the drop, Q is the net electric charge, and σ is the surface energy density of the boundary interface, then Rayleigh's criterion for morphological instability is

$$Ra \equiv \frac{Q^2}{16\pi\sigma R^3} > 1 \ , \tag{1}$$

which can be rewritten as

$$\frac{\tau^2}{4\sigma R} > 1 \ , \tag{2}$$

where τ is the surface charge density.

Per the Rayleigh criterion (Eq. 1), as long as the total charge *Q* or the surface tension σ is large, the surface tension influence dominates, and the spherical shape appears to be stable with respect to small nonspherical perturbations of the shape. But when the dimensionless number Ra exceeds the critical value $Ra_{crit} = 1$, the spherical shape becomes unstable and yields the spheroidal shape. For example, a drop of water 1 mm in radius is unstable if it is charged to several volts.

Rayleigh's original study was picked up by many researchers including Tonks,[9] Frenkel,[10] Melcher and Taylor,[11] and others (see extensive lists of relevant references in Rosensweig[12]). Their research showed that, in general, as in Rayleigh's original work, the presence of sufficiently large electrostatic or magnetostatic fields leads to the morphological destabilization of plane interfaces. This class of instabilities plays a role in the physics of vacuum discharges and in the operation of liquid metallic emitters of charged particles. These phenomena continue to be in a state of active research. One of the primary questions of interest is that of the equilibrium shape of destabilized interfaces.[13] From the mathematical point of view, such problems of equilibrium are challenging, deeply nonlinear problems with unknown boundaries, reminiscent of the classical problem of the equilibrium shape of a crystal.

3. Algorithm of Computer Modeling of Equilibrium Shape and Poststable Evolution

3.1 Energy Minimization Approach

The Rayleigh problem can be treated as the problem of minimization of the total energy W comprising 2 ingredients, the total electrostatic energy E_{elec} and the total surface energy E_{surf}:

$$W = E_{elec} + E_{surf} \ . \tag{3}$$

The surface energy density per unit area in solid substances or in liquid crystals is characterized by its dependence on the normal $\vec{N}$: $\sigma\left(\vec{N}\right)$. For isotropic liquid we have just a positive constant function $\sigma\left(\vec{N}\right) = \sigma = const$.

The total electrostatic energy depends upon the charge loading mechanism. In choosing the charge loading mechanism, we first supply a certain amount of the electric charge *Q* and then disconnect the battery. Then we arrive at the following expression of the total energy ingredients:

$$E_{elec} = \frac{1}{8\pi}\int\limits_{Space} d\Omega|\nabla\varphi|^2, \quad E_{surf} = \sigma\int\limits_{\Xi} d\Xi. \tag{4}$$

The second integral in Eq. 4 is the total area of the conductor's boundary Ξ.

There are actually 2 functional degrees of freedom in the minimization problem: 1) the unknown surface distribution of electric charge τ and 2) the position of the boundary Ξ.

In this minimization problem, the condition of mass conservation should be taken into account:

$$\int\limits_{\Xi} d\Xi z^i N_i = 3V_0 = const \ , \tag{5}$$

where N_i is the unit normal to the boundary.

Also, given the total electric charge *Q* of the conductor, we have to respect the constraint:

$$\int\limits_{\Xi} d\Xi\,\tau = Q = const \ . \tag{6}$$

It should be taken into account that the distribution of electric charge τ appears in the total energy implicitly. Indeed, given the probe surface charge distribution τ and the interface position, we have to calculate the corresponding electrostatic energy E_{elec}. To do this, we have to calculate the electrostatic potential φ. In other words, one has to solve the electrostatic problem for that geometry and that surface charge distribution. That goal can be accomplished by calculating the singular integral

$$\varphi\left(\vec{z}\right) = \int\limits_{\Xi} d\Xi^* \frac{\tau\left(z^*\right)}{\left|\vec{z}-\vec{z}^*\right|} \ , \tag{7}$$

where $\vec{z}^*$ is the radius vector of the points on the boundary. The same goal can be achieved or by solving the boundary value problem of electrostatics:

1) inside conductor

$$\varphi = \varphi_{in} = const\ . \tag{8}$$

2) outside conductor

$$\nabla^i \nabla_i \varphi = 0\ . \tag{9}$$

3) at the interface liquid-vacuum

$$\varphi_{in} = \varphi_{out},\ \nabla_i \varphi_{out} N^i = 4\pi\tau\ . \tag{10}$$

4) at infinity

$$\varphi \to 0\ . \tag{11}$$

Thus, from the mathematical point of view, we arrive at the problem of multidimensional calculus of variations with the isoperimetric and differential constraints, and with the unknown boundary.

3.2 Two-Step Algorithm of the Energy Minimization

Now we can formulate the suggested 2-step numerical algorithm. Let Ξ, τ, B, and W_{tot} be the current positing of the boundary and the current values of the surface charge density, the volume of the drop, and the total energy of the system, respectively. We use the notations Ξ^*, τ^*, B^*, and W_{tot}^* for the same quantities in the updated position.

The suggested algorithm of updating provides the validity of 2 following relationships:

$$B^* \cong B \tag{12}$$

and

$$W_{tot}^* \le W_{tot}\ . \tag{13}$$

The second step requires, in addition to the current value of τ, the calculation of the current mean curvature R of the boundary. Using current values of τ and R, the algorithm requires the calculation of the function χ,

$$\chi = 2\pi\tau^2 - \frac{\sigma}{R} \tag{14}$$

at each point of the current boundary Ξ.

To get the updated position of the conductor's boundary, move each point of the surface in the direction of the local unit normal $\vec{N}$. The distances Δ of displacement should be different in different points of the boundary (Fig. 3); it should be calculated in accordance with the relationship

$$\Delta = -\varepsilon\left(\chi - \chi_{mean}\right), \tag{15}$$

where ε is a sufficiently small positive number and χ_{mean} is the mean value of the of the function χ over the surface Ξ:

$$\chi_{mean} = \frac{1}{area\,\Xi}\int_{\Xi} d\Omega \ . \tag{16}$$

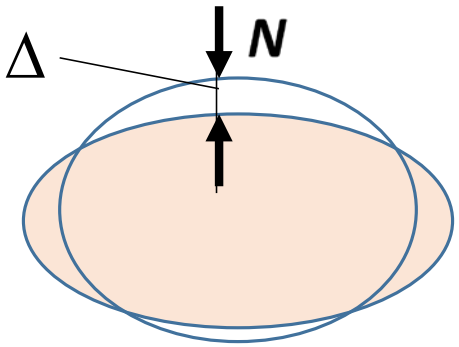

Fig. 3 Toward the 2-stage algorithm of computational analysis of the Rayleigh instability

We call this surface the upgraded surface Ξ^* and the domain B^* inside it the upgraded domain. Then, we solve the electrostatic problem[4,6–9] for the upgraded domain and calculate the upgraded total energy W^*_{total}.

4. Conclusion

We presented an algorithm allowing one to explore numerically the Rayleigh problem dealing with destabilization of an electrically charged incompressible liquid conductor. The algorithm is based on the variational formulation of the

Rayleigh problem, and, conceptually, it allows one to explore the under-critical, critical, and super-critical regimes of charging. Also, the method permits further extension for polarizable substances.

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